4.6 – Dimension

Due Sun

Theorem 4.6.2 Let V be a finite-dimensional vector space, and let $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ be any basis for V.

1. If a set in V has more than n vectors, then it is linearly dependent.
2. If a set in V has fewer than n vectors, then it does not span V.

The proof essentially counting variables and equations in a linear system.

Theorem 4.6.1 All bases for a finite-dimensional vector space have the same number of vectors.

Definition: The **dimension** of a finite-dimensional vector space V is denoted by $\dim(V)$ and is defined to be the number of vectors in a basis for V (in some physical contexts, "dimension" is referred to as **degrees of freedom**). In addition, the zero vector space is defined to have dimension zero.

basis: $\emptyset$

Find a basis for the solution space of each homogeneous linear system, and find the dimension of that space.

#1

$$\begin{aligned} x_1 + x_2 - x_3 &= 0 \\ -2x_1 - x_2 + 2x_3 &= 0 \\ -x_1 \qquad + x_3 &= 0 \end{aligned}$$

$$\begin{bmatrix} 1 & 1 & -1 \\ -2 & -1 & 1 \\ -1 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$R_1: \ x_1 = x_3$

$R_2: \ x_2 = 0$

$$\vec{x} = \begin{bmatrix} t \\ 0 \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} t$$

Basis for solution space $\left\{\begin{bmatrix}1\\0\\1\end{bmatrix}\right\}$.

Dimension: 1

#5 $2x_1 - 6x_2 + 2x_3 = 0$

$3x_1 - 9x_2 + 3x_3 = 0$

$$\begin{bmatrix}2 & -6 & 2\\3 & -9 & 3\end{bmatrix} \to \begin{bmatrix}1 & -3 & 1\\0 & 0 & 0\end{bmatrix}$$

$$x_1 = 3x_2 - x_3$$

$$\vec{x} = \begin{bmatrix}3s-t\\s\\t\end{bmatrix} = \begin{bmatrix}3\\1\\0\end{bmatrix}s + \begin{bmatrix}-1\\0\\1\end{bmatrix}t$$

Basis: $\left\{\begin{bmatrix}3\\1\\0\end{bmatrix}, \begin{bmatrix}-1\\0\\1\end{bmatrix}\right\}$, Dimension: 2

#8 In each part, find a basis for the given subspace of R^4, and state its dimension.

a. All vectors of the form $(a,\ b,\ c,\ 0)$.

b. All vectors of the form $(a,\ b,\ c,\ d)$, where $d = a + b$ and $c = a - b$.

c. All vectors of the form $(a,\ b,\ c,\ d)$, where $a = b = c = d$.

a) $\{(1,0,0,0), (0,1,0,0), (0,0,1,0)\}$, Dim: 3

b) $$\begin{bmatrix}a\\b\\c\\d\end{bmatrix} = \begin{bmatrix}a\\b\\a-b\\a+b\end{bmatrix} = \begin{bmatrix}1\\0\\1\\1\end{bmatrix}a + \begin{bmatrix}0\\1\\-1\\1\end{bmatrix}b$$

Basis: $\{(1,0,1,1), (0,1,-1,1)\}$, Dim: 2

c) Basis: $\{(1,1,1,1)\}$, Dim: 1

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

#9 Find the dimension of each of the following vector spaces.
a. The vector space of all diagonal $n \times n$ matrices.
b. The vector space of all symmetric $n \times n$ matrices.
c. The vector space of all upper triangular $n \times n$ matrices.

a) $D = \begin{bmatrix} d_1 & & 0 \\ & d_2 \ddots & \\ 0 & & d_n \end{bmatrix}$ $n \times n$ diag. matrices have up to n nonzero entries $\Rightarrow$ dim is n.

(b) & (c) have the same dimension, as the free entries for both exist on and above the main diagonal.

Dim: $n + (n-1) + (n-2) + \cdots + 3 + 2 + 1$

$$= \sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

Theorem 4.6.4 Let V be an n-dimensional vector space, and let S be a set in V with exactly n vectors. Then S is a basis for V if and only if S spans V or S is linearly independent.

$(1,2,3), (2,4,6), (3,6,9)$

That is:
- correct # of vectors (dim)
- spanning
- lin. indep.

Any 2 of these guarantees the third.

Theorem 4.6.5 Let S be a finite set of vectors in a finite-dimensional vector space V.
a) If S spans V but is not a basis for V, then S can be reduced to a basis for V by removing appropriate vectors from S.
b) If S is a linearly independent set that is not already a basis for V, then S can be enlarged to a basis for V by inserting appropriate vectors into S.

$\vec{e}_1, \vec{e}_2, \vec{e}_3, \vec{e}_4$

#13 Find standard basis vectors for R^4 that can be added to the set $\{\mathbf{v}_1, \mathbf{v}_2\}$ to produce a basis for R^4.
$\mathbf{v}_1 = (1,\ -4,\ 2,\ -3)$, $\mathbf{v}_2 = (-3,\ 8,\ -4,\ 6)$

$$\begin{bmatrix} 1 & -3 & 1 & 0 & 0 & 0 \\ -4 & 8 & 0 & 1 & 0 & 0 \\ 2 & -4 & 0 & 0 & 1 & 0 \\ -3 & 6 & 0 & 0 & 0 & 1 \end{bmatrix} \to \begin{array}{c} \begin{array}{cccccc} \vec{v}_1 & \vec{v}_2 & \vec{e}_1 & \vec{e}_2 & \vec{e}_3 & \vec{e}_4 \end{array} \\ \begin{bmatrix} 1 & 0 & -2 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 & 0 & -1/3 \\ 0 & 0 & 0 & 1 & 0 & -4/3 \\ 0 & 0 & 0 & 0 & 1 & 2/3 \end{bmatrix} \end{array}$$

using pivots

$\{\vec{v}_1, \vec{v}_2, \vec{e}_2, \vec{e}_3\}$ because the matrix with these vecors as its columns reduce to the identity matrix.

#17 Find a basis for the subspace of R^3 that is spanned by the vectors
$\mathbf{v}_1 = (1,\ 0,\ 0)$, $\mathbf{v}_2 = (1,\ 0,\ 1)$, $\mathbf{v}_3 = (2,\ 0,\ 1)$, $\mathbf{v}_4 = (0,\ 0,\ -1)$.

$$\begin{bmatrix} 1 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & -1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\uparrow \ \uparrow$ using pivots

The desired basis is $\{\vec{v}_1, \vec{v}_2\}$.

#20 In each part, let $\boxed{T_A}$ be multiplication by A and find the dimension of the subspace of R^4 consisting of all $\boxed{\text{vectors } x \text{ for which } T_A(\mathbf{x}) = \mathbf{0}}$.

kernel of T_A

a. $\begin{bmatrix} 1 & 0 & 2 & -1 \\ -1 & 4 & 0 & 0 \end{bmatrix}$

b. $\begin{bmatrix} 0 & 0 & 1 & 1 \\ -1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}$

This is asking for the dimension of the kernel.

a. $\begin{bmatrix} 1 & 0 & 2 & -1 \\ -1 & 4 & 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 2 & -1 \\ 0 & 4 & 2 & -1 \end{bmatrix}$

$\to \begin{bmatrix} 1 & 0 & 2 & -1 \\ 0 & 1 & 1/2 & -1/4 \end{bmatrix} \Rightarrow \begin{aligned} x_1 &= -2x_3 + x_4 \\ x_2 &= -\tfrac{1}{2}x_3 + \tfrac{1}{4}x_4 \end{aligned}$

$$\vec{x} = \begin{bmatrix} -2s + t \\ -\frac{1}{2}s + \frac{1}{4}t \\ s \\ t \end{bmatrix} = \begin{bmatrix} -2 \\ -\frac{1}{2} \\ 1 \\ 0 \end{bmatrix} s + \begin{bmatrix} 1 \\ \frac{1}{4} \\ 0 \\ 1 \end{bmatrix} t$$

Basis for kernel has 2 vectors $\Rightarrow \dim(\ker(T_A)) = 2$.

$A\vec{x}=\vec{0}$	$A\vec{x}=\vec{0}$	$T_A(\vec{x})=\vec{0}$
Matrix	System	Transformation
null(A)	Solution space	Kernel